The size scaling of a gravitational granular instability, analogous to the hydrodynamic Rayleigh-Taylor instability, is investigated experimentally, numerically, and theoretically. The flow of grains and an interstitial gas is studied in a closed, vertical Hele-Shaw cell, where the growth of unstable bubbles and fingers is characterized as the grains fall. The dependence of these structures on the system and grain sizes is investigated. As the size of the grains increases, the characteristic inverse length scale of the structures increase accordingly, leaving the dimensionless product of the inverse length scale \( \langle k \rangle \) and the grain diameter \( d \) invariant. Plotting the rescaled quantity \( d\langle k \rangle \) as function of time yields numerical and experimental data-collapses for diameters ranging from 70 µm to 570 µm. A theoretical interpretation of the invariance, based on the scaling properties of the model equations, suggests a gradual breakdown of the invariance as the size of the grains deviates significantly from ~200 µm.

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Keywords: granular flow, gravitational instability, Rayleigh-Taylor instability, disordered media

I. INTRODUCTION

Granular materials are the basis of abundant industrial and natural processes, and an integral part of our everyday life. Dry granular flows have been widely studied over the past twenty years [1], but the study of granular flows where the interstitial fluid plays an important role is still in its infancy. Many natural phenomena pertaining to such granular/liquid flows are active topics of current research, e.g., sedimentation [2, 3], erosion and river evolution [4], underwater avalanches and turbidites [5], and soil fluidization during earthquakes [6]. Industrial processes, such as pneumatic transport, fluidized beds, catalytic cracking [1, 7, 8] would also benefit from advances in granular/fluid flow research.

The granular Rayleigh-Taylor instability arises when a closed, vertical Hele-Shaw cell, partially filled with fine grains and air at atmospheric pressure, is rapidly rotated to bring the dense packing of grains above the supernatant air. During the rotation the grains and gas pass each others in layers, in analogy with the granular Boycott effect [9, 10], and no fingers are formed. Just before the cell is vertical this laminar flow ceases and an approximately flat front is defined by the falling grains. From this front fine fingers emerge that subsequently develop into coarser finger-bubble structures. In contrast to the classical Rayleigh-Taylor instability [11] the coarsening process observed in the granular case will almost immediately bring about a reduction in the number of fingers, and a corresponding increase in the size of the bubbles. New fingers will then form in the centre of the rising bubbles as they reach a certain width, and the characteristic size of the structures is thus maintained at a stable value. The granular instability is previously studied both numerically and experimentally [12], using the same model and setup as in the present Paper, with the conclusion that the two competing mechanisms, one producing finer scales, the other producing coarser scales, are well reproduced by the numerical model.

The purpose of the present investigation is to study the variations of the flow structures under change of spatial scale of both these grains and the Hele-Shaw cell. Size invariance is not commonly studied for granular materials. Indeed, granular flows are quite sensitive to initial preparation and external perturbations, so that systematic grain and system size change are delicate studies. In the present Paper, using the complementarity of experimental and numerical techniques, we perform a systematic study of the arising structures in the granular Rayleigh-Taylor instability, under a rescaling of the system and grain sizes.

The experimental, numerical, and theoretical results presented all indicate a size invariance of the structures formed by the instability. However, the theoretical analysis predicts a breaking of the invariance if the grains are too small or too big. The simulations use grains ranging from 70 µm to 490 µm in diameter, and the experiments use grains ranging from 80 µm to 570 µm in diameter. Consistent data collapses are obtained from these results and demonstrate a fairly robust invariance within the given limits.

The numerical model uses a hybrid technique that affords a continuum description of the air and a discrete description of the granular phase [13, 14]. In order to make the simulations computationally affordable and the modelling as simple as possible, particle friction and the space coordinate perpendicular to the cell are ignored in the model. In addition, the pressure field of the air is described at a scale greater than the diameter of the grains. We stress that while in Ref.[12] we applied simulations in order to match the experiments as closely as possible, the simulations in the present work plays a role that is more complementary to the experiments. In particular, we carry out the simulations and experiments in somewhat different parameter regimes.

The paper is organized as follows. A presentation of the
and a layer of silicone paste is applied along the outer edge of pressure by clamps to make the adhesive tape attach firmly, between the glass plates and the silicone: The cell is put under a silicone frame. Great care is taken to ensure an air tight coupling be-
tween the grains, and the upper surface of the beads is smoothed out is then flipped a few times to yield a random loose packing of the grains due to cohesion arising from capillary bridges or electrostatic forces.

In total four cells are assembled and filled with monodisperse polystyrene beads of respectively 80 µm, 140 µm, 230 µm and 570 µm in diameter. The inner dimensions of the cells, given in Tab. I, are designed according to the diameters of the grains they contain; the cells scale in all three directions in order to keep the number of grains in the cells approximately constant. The experimental cells are somewhat taller than the numerical cells to account for the grains that settle during the rotation. The horizontal filling of the cells makes it difficult to control the filling fraction, and it may vary between the four cells.

Dynoseeds polystyrene spheres of density 1.05 g/cm³, manufactured by Microbeads [27], are used in the experiments. During the assembling of the cells the TS 500-53 beads was presumed to have a diameter of 500 µm. However, an analysis performed by Microbeads shows that the mean diameter of these grains is in fact 570 µm. This explains the slight discrepancy between the width of the cell and the diameter of the grains in this case.

The experiments performed with grains of 80 µm and 140 µm in diameter are more challenging due to the small length scales involved, and the larger sensitivity of these systems to the impact with the stopping bar. For the bigger cells it is possible to soften the impact by slowing down the cell just before it hits the bar, and the patterns are also less affected by the induced shock.

The numerical model has proved to be very consistent in reproducing experimentally observed structures in granular flows and instabilities in the regime of low Reynolds numbers [13–16]. The theoretical basis and derivation of the model are given in Refs. [13, 14] and a description of the current imple-

<table>
<thead>
<tr>
<th>diameter (µm)</th>
<th>80</th>
<th>140</th>
<th>230</th>
<th>570</th>
</tr>
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<tr>
<td>width (mm)</td>
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<td>91</td>
<td>200</td>
</tr>
<tr>
<td>height (mm)</td>
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<td>86</td>
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<td>305</td>
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<tr>
<td>depth (mm)</td>
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<td>1.7</td>
<td>2.3</td>
<td>5.4</td>
</tr>
</tbody>
</table>

Table I: Listing of the diameters of the grains and the dimensions of the Hele-Shaw cells used in the experiments.
The model acknowledges the particularity of the granular phase by describing it as a deformable porous material made up by discrete, rigid particles. The dynamics of each grain is governed by Newton's second law

$$\frac{dv}{dt} = mg + F_I - \rho_n \nabla P,$$  \hspace{1cm} (1)

where $m$ is the grain mass, $v$ is the grain velocity, $F_I$ is the inter-particle force normal to their contact plane which keeps the grains from overlapping, $\rho_n = \rho g / m$ is the number density, $\rho g$ is the mass density of the material which the grains are made of, $\rho$ is the solid fraction, and $\nabla P$ is the local pressure gradient. Contact dynamics [17–20] calculate the distribution of the inter-particle forces $F_I$ by solving iteratively the set of kinematic constraints imposed by the non-overlapping, rigid grains in contact. The flow of air through the packing is described by Darcy’s law [21] with a local permeability $\kappa$ given by the empirical Carman-Kozeny relation [22] which is applicable to a dense collection of spheres. Together with mass conservation for the grains and air, and the isothermal gas law, this leads to the gas equation [13, 14]

$$\phi \left( \frac{\partial P}{\partial t} + u \cdot \nabla P \right) = \nabla \cdot \left( \mu \frac{\rho}{\kappa} \nabla P \right) - P \nabla \cdot u,$$ \hspace{1cm} (2)

where $\phi = 1 - \rho$ is the porosity, $\rho$ is the solid fraction, $\kappa$ is the permeability, and $u$ is the velocity field of the granular phase. $P$ is the gas pressure, and $\mu$ is the gas viscosity.

The continuous solid fraction and velocity fields on the grid, $\rho(x, y)$ and $u(x, y)$, are calculated from the masses $m_i$ and velocities $v_i$ of the individual grains (with index $i$) by a linear smoothing function $s(r - r_0)$ expressed mathematically as [13]

$$s(r - r_0) = \begin{cases} 
(1 - \frac{\Delta x}{l})(1 - \frac{\Delta y}{l}) & \text{if } \Delta x, \Delta y < l \\
0 & \text{otherwise}, 
\end{cases} \hspace{1cm} (3)
$$

where $r(x, y)$ is the position of the grain, $r_0(x_0, y_0)$ is the position of the grid node, $\Delta x = |x - x_0|$ and $\Delta y = |y - y_0|$ are the relative distances, $l$ is the lattice constant, and $\sum_k s(r - r_k) = 1$ with $k$ indexing the four neighbouring sites of the grain positioned at $r$.

The Carman-Kozeny relation is not valid for solid fractions less than 0.25 [24], and a cutoff on the solid fraction is thus introduced. As a consequence the porosity in the very dilute regions of the system is underestimated, giving rise to slightly increased pressure gradients which tend to distort the motion of the fingers. The effect of this artifact is however decreasing with increasing granular inertia. The dissipation in the system is governed by a coefficient of restitution that controls the loss of energy associated with each collision. In order to have the solid fraction calculated in a 2D system applicable to a collection of spheres, the original 2D solid fraction is multiplied by $2/3$, which is the solid fraction ratio of randomly close packed spheres to randomly close packed disks. These matters are elaborated in Sec. II C of Ref. [13].

A series of seven simulations are performed, and each simulation uses monodisperse grains of different diameters, see Tab. II. The same relative start configuration of 160000 grains is used in all the simulations, and the size of the system, given in Tab. II, scales proportionally to the diameters of the grains. The mass density of the grains is 2.5 g/cm$^3$, a common value for glass beads. In the simulations we have introduced a larger density than in the experiments in order to minimize the numerical artifacts, manifested as slightly curved and buckled fingers, that become visible as the inertia of the grains decreases. This behaviour might be explained by the increased importance of gas inertia, and the cutoff in the solid fraction which leads to overestimated forces in the dilute regions.

**IV. RESULTS AND ANALYSIS**

Images from each of the four experiments and seven simulations are respectively shown in Figs. 2 and 3, and the corresponding widths of the cells are given in Tabs. I and II.

From these images it is evident that the dynamics of the instability and the structures that appear depend strongly on the size of the beads. The interface of grains separating the upper part of the packing from the air immediately below is sharply defined for grains of diameters less than 200 μm. For diameters greater than 200 μm, small bubbles appear in the bulk of the packing, above the interface, and horizontal structures are formed in addition to the vertical fingers. The horizontal filaments, and the precursory bubbles separating them, increase in number as the grain diameter increases. These structures are analogous to the bubbles observed in the ripple instability arising in a tilted tube of sand [10, 15].

While these overall features are observed both experimentally and numerically, differences still exist. However, since the densities of the grains used in the experiments and simulations are different, it is not expected that the experimental and

<table>
<thead>
<tr>
<th>diameter (μm)</th>
<th>70</th>
<th>140</th>
<th>210</th>
<th>280</th>
<th>350</th>
<th>420</th>
<th>490</th>
</tr>
</thead>
<tbody>
<tr>
<td>width (mm)</td>
<td>28</td>
<td>56</td>
<td>84</td>
<td>112</td>
<td>140</td>
<td>168</td>
<td>196</td>
</tr>
<tr>
<td>height (mm)</td>
<td>34</td>
<td>68</td>
<td>102</td>
<td>136</td>
<td>170</td>
<td>204</td>
<td>238</td>
</tr>
</tbody>
</table>

Table II: Listing of the diameters of the grains and the dimensions of the cells used in the simulations.
numerical results are directly comparable. The motivation for this investigation is to study the size scaling of the experimental and numerical structures separately. Moreover, the height and filling fraction of the experimental and numerical cells are different. This explains the variation in relaxation times, i.e. the time it takes to reach equilibrium, and the fact that bubbles fade away before they reach the surface for some of the simulations but not for the experiments. While the bubbles in the experiments do not fade away entirely, there is still a decrease in the average height of the decompacted region, i.e. the total volume of the structures, as the grain diameters increase.

A. Characteristic inverse length scale

A measurement of the time dependent, horizontal size of the structures is provided by means of Fourier transforms of the solid fraction field in the simulations, and of the image pixel values in the experiments. The procedure is as follows.

The power spectrum $S_j(k)$ of each horizontal line $j$ of the solid fraction grid is obtained by the discrete Fourier transform [25], using Hamming data-windows to avoid frequency leakage due to the non-periodic character of the system. An average over these power spectra yields a single power spectrum $S(k)$ describing the distribution of amplitudes of the Fourier modes as function of the horizontal wave numbers $k$. 

Figure 2: A series of images from experiments where monodisperse polystyrene beads with diameters ranging from 80 µm to 570 µm are used.

Figure 3: A series of snapshots from simulation using glass beads with diameters in the range of 70 µm to 490 µm. The same packing is used in all simulations and the size of the system scales with the diameter of the grains.
in the system at a given time. From this distribution the mean wave number \( \langle k \rangle \) and the standard deviation \( \sigma_k \) are obtained by the usual definitions

\[
\langle k \rangle = \frac{\sum k \bar{S}(k)}{\sum S(k)} \quad \text{(4)}
\]

\[
\sigma_k = \sqrt{\frac{\sum k^2 \bar{S}(k)}{\sum S(k)} - \langle k \rangle^2} \quad \text{(5)}
\]

where the summation is over all \( k \)-values. To avoid contributions in \( \bar{S}(k) \) from the upper surface of the packing, the power spectra \( S_k(\langle k \rangle) \) calculated above or at the surface are excluded from \( \bar{S}(k) \). The mean wave number \( \langle k \rangle \) should be a measure of the structures formed along the base or in the bulk of the packing, not the profile of the upper surface.

The same procedure is followed to calculate \( \langle k \rangle \) and \( \sigma_k \) for the experimental data, but with the solid fraction field replaced by the values of the image pixels. The gray scale pixel value is assumed to be linearly related to the solid density. The inaccuracy of this assumption is not likely to have a large effect on the measurements of \( \langle k \rangle \) and \( \sigma_k \). *Eirik removes: The pixel values \( c \) range from 0 (black) to 255 (white), and are inversely related to the solid fraction, i.e. pixels in dilute, white regions of the experimental images have high pixel values, whereas the solid fraction in these regions is low. Nonetheless, if the dependence of the pixel value on the solid fraction is well approximated in the experiments by a linear relationship \( c = a - b \rho \) with \( a \) and \( b \) two constants, using the pixel gray value or the solid fraction should lead to the same average wavenumber \( \langle k \rangle \) and standard deviation \( \sigma_k \). The number of pixels across the width of the experimental images should be the same, in order to have comparable power spectra, and the images are rescaled by bicubic interpolation [26] to a common width determined by the width of the smallest image. This is not necessary for the numerical data since the solid fraction grid has the same number of nodes for all the simulations.*

Figs. 4a and 4b shows respectively the temporal evolution of \( \langle k \rangle \) and \( \sigma_k \) for the seven simulations. The dimensionless quantities \( d\langle k \rangle \) and \( d\sigma_k \) are calculated by rescaling the vertical axes of these plots by the diameter \( d \) of the grains, and the data-collapses respectively shown in Figs. 5a and 5b are obtained. The initially decreasing \( d\langle k \rangle \) in Fig. 5a represents the coarsening observed in the numerical images of Fig. 3. After about 0.2 seconds the systems collectively reach a steady state where the size of the structures is approximately the same when measured in units of grain diameters. This size invariance is also apparent in the images of Fig. 3. However, the invariance of the 490 µm data is broken as the curve deflects just before \( t = 0.3 \) seconds in Figs. 5a and 5b. The subsequent divergences of the remaining simulations follow in decreasing order of diameter. This divergence is the signature of bubbles...
disappearing, either at the upper surface \((d = 80 \mu m \text{–} 350 \mu m)\), or in the bulk of the packing \((d = 420 \mu m \text{–} 490 \mu m)\), and is a result of the permeability increasing as the square of the grain diameter [22]. With a larger reservoir of air the bubbles would propagate to the surface. The characteristic wave number of the relaxed system, having no macroscopic structure, is given by the intrinsic heterogeneity of the granular packing, and this explains the rapid increase of \(d(k)\) when the bubbles disappear. This is equivalent to the situation in the very beginning of the instability before any fingers were formed.

An explanation for the fading bubbles is suggested in the following. Provided that the diameter of the grains scales as \(d \rightarrow \lambda d\) and the dimensions of the cell scales as \(L \rightarrow \lambda L\), the permeability will scale as \(\kappa \rightarrow \lambda^2 \kappa\), by the Carman-Kozeny relation [22], while the viscosity and the pressure gradient are invariant. Then, by Darcy’s law [21], the sedimentation velocity of the air through the packing scales as \(u_0 \rightarrow \lambda^2 u_0\). The time \(\tau\) it takes to transport a volume \(V\) of air at velocity \(u_0\) through a permeable area of cross section \(A\) is given by \(\tau = V/(u_0 A)\), and scales as \(\tau \rightarrow \lambda^{-1} \tau\). Hence, the bubbles will fade away at an increasing rate, and for a given grain diameter and filling height the bubbles will disappear before they reach the surface. This relation does not consider the volume of air transported by the bubbles, and is only valid for larger grains when the flux of air through the pores in the packing is greater than the volume of air transported by the bubbles in a given time.

The data-collapse for \(d(k)\) in Fig. 5a is quite conclusive for times less than 0.3 seconds, except for the 70 \(\mu m\) data which falls somewhat short of the collapse. This behaviour might be explained by the scaling properties of the model equations discussed in Sec. V, where the conclusion is that the size invariance is not valid for very small or very large grains.

Data-collapses obtained from the experimental data are respectively shown in Figs. 6a and 6b for \(d(k)\) and \(d \sigma_k\). The inset plots shows the temporal evolution of the unscaled quantities \(\langle k \rangle\) and \(\sigma_k\). For each diameter an average over three experiments is performed for the four different grain diameters \(d\), given in units of \(\mu m\) in the legend boxes. The inset plots shows the evolution of \(\langle k \rangle\) and \(\sigma_k\). The standard deviation is given by the error bars.

Data-collapses of the experimental data are respectively shown in Figs. 6a and 6b for \(d(k)\) and \(d \sigma_k\). The inset plots shows the temporal evolution of the unscaled quantities \(\langle k \rangle\) and \(\sigma_k\). For each diameter an average over three experiments is performed for the four different grain diameters \(d\), given in units of \(\mu m\) in the legend boxes. The inset plots shows the evolution of \(\langle k \rangle\) and \(\sigma_k\). The standard deviation is given by the error bars.

Grains to be quite far from horizontal. The shock also leaves a signature on the evolution of \(\langle k \rangle\) which is used to define time zero: In response to the erased fingers \(\langle k \rangle\) will increase. The subsequent reemergence of fingers is followed by a decrease in \(\langle k \rangle\), and this is used to define time zero in the plots of Fig. 6.

**B. Wave number growth rates**

If the dynamical equations controlling the system can be initially linearized to a good approximation, then the physical quantities, represented in vector form as \(\mathbf{A}(t)\), follow initially an approximate law of the type \(\mathbf{A}'(t) = \mathbf{M} \cdot \mathbf{A}(t)\). The initial behaviour should thus be described by

\[
\mathbf{A}(t) = \text{Re}(\mathbf{A}(0) \cdot \exp(\mathbf{M} t)) .
\]  

Eirik removes: If furthermore the Fourier modes are eigenvectors of the evolution matrix \(\mathbf{M}\), as is the case in many periodic or finite problems, then The eigenvalue associated to the k-Fourier mode is denoted \(\alpha_k + i\omega_k\), and the k Fourier modes behave as

\[
\tilde{\rho}_k(t) \propto \tilde{\rho}_0 \exp(i(\alpha_k t + \omega_k t)) ,
\]
Figure 7: Wave numbers of the solid fraction power spectra plotted semi-logarithmically as functions of time to show the growth of the different modes in the $d = 140 \, \mu m$, $280 \, \mu m$ and $490 \, \mu m$ simulations. Only a subset of wave numbers is shown.

Figure 8: Dispersion relations valid for $t < 0.1$ seconds and $d = 140 \, \mu m$, $280 \, \mu m$ and $490 \, \mu m$. The growth rates are obtained by linear fits over the approximately linear sections indicated by vertical lines in Fig. 7. An average over 8 wave numbers is performed, and the standard deviation is given by the error bars.

where $\alpha_k$ is called a growth rate. The power spectrum should then behave as

$$S(k, t) = |\tilde{\rho}_k(t)|^2 = S(k, 0) e^{2\alpha_k t}. \quad (8)$$

We look for possible behavior of this type in the numerical data.

In addition to calculating the mean and standard deviation of $S(k)$, the power spectrum amplitudes of the individual $k$-modes in $S(k)$ are plotted semi-logarithmically as functions of time to bring out exponentially growing modes as linear. Such plots are shown in Figs. 7a, 7b, and 7c where the growth rates for six different wave numbers are plotted for simulations using grains of respectively $140 \, \mu m$, $280 \, \mu m$, and $490 \, \mu m$ in diameter. The wave numbers are rescaled by the diameter $d$. The following features are emphasized: The rather non-linear growth of the wave numbers in Fig. 7a is modified as the diameter increases, and the growth in Fig. 7c is quite close to linear in the limited region marked by the vertical lines. After the approximate linear growth, i.e. after about 0.1 seconds, the wave numbers take on a fluctuating behaviour where the wave length seems to increase with the diameter. These fluctuations might be related to the two competing mechanisms of
the instability: The coarsening process merging fingers, and the
refining process producing fingers.

Despite the imperfect linearity, the growth rates \( \alpha_k \) of all
the wave numbers in the power spectrum \( S(k) \) are obtained by
linear least squares fits over the regions marked by vertical
lines in the plots of Fig. 7. The dispersion relations, shown
in Figs. 8a, 8b, and 8c for grains of respectively 140 \( \mu m \), 280
\( \mu m \), and 490 \( \mu m \) in diameter, are calculated by averaging over
eight wave numbers, and the standard deviation is given by the
error bars. The horizontal axes of these relations are collapsed
by multiplying \( d \) and \( k \), and as the diameter increases a peak
centered around \( \Delta k = 0.02 \) emerges. The average \( \alpha_k \) value
is however decreasing as the diameter increases.

It is important to point out that these dispersion relations are
only valid for small times, i.e. \( t < 0.1 \) s. The final dominating
wave length can not be predicted from these relations since
non-linear effects take over and govern the evolution of the
final structures.

Although the system behaves approximately linearly for
many wavelengths at early times \( (t < 0.1 \) s), as shown in Fig. 8,
the fastest growing mode does not necessarily coincide with
the one selected as dominating in the structure. Indeed, the
power spectrum \( S(k, t) \propto S(k, 0) e^{2\alpha_k t} \), depends both on the
initial state \( S(k, 0) \) and on the values of the growth rates \( \alpha_k \).
Since the approximate linear behavior is only valid for short
times, the exponential dependence on time does not dominate
the behavior of maximum \( S(k, t) \) and is not responsible for
the selected wave number of the final structures. The maxi-
imum of the power spectrum \( S(k, t) \) shifts between frequen-
cies as the size of the structures increases, which may result
from an interplay in the multiplication of \( S(k, 0) \) and \( e^{2\alpha_k t} \)
since their maxima as function of \( k \) (at a given time \( t \)) do not
coincide. This allows to interpret the observed shift of charac-
teristic wavelengths in Figs. 4, 5 and 6, even for early times
\( (t < 0.1 \) s) when the system behaves approximately linearly.

V. THEORETICAL INTERPRETATION

Let us assume that we have a solution of our gas- and grain
equations. Now, if we magnify all physical scales, the veloc-
ities and the pressure variations by a constant factor \( \lambda \), how
close will we come to a solution of our equations?

In mathematical terms, let \( P, \rho(x, t) = m \rho_n(x, t) \) and \( v_i \)
be the pressure field, the mass density field and particle veloc-
ities that solve the equations

\[
\frac{\phi}{\beta} \frac{\partial P}{\partial t} = \nabla \cdot \left( \frac{\kappa P \nabla P}{\rho} \right) - P \nabla \cdot \mathbf{u}
\]

\[
\frac{m \partial v}{\partial t} = mg + F - \frac{m \nabla P}{\rho},
\]

where as before \( \mathbf{u} \) is just the local average of the \( v_i \)'s. We
split the velocity as follows: \( v_i = \delta v_i + \mathbf{u}_0 \) and \( \mathbf{u} = \delta \mathbf{u} +
\mathbf{u}_0 \), where \( \mathbf{u}_0 \) is the constant sedimentation velocity of a close
packed system. We substitute \( P = P_0 + \delta P \) and \( F = ma \)
and make the observation that \( \delta P \ll P_0 \) which leads to the
justified approximation

\[
\frac{\phi}{\beta} \frac{\partial \delta P}{\partial t} = \nabla \cdot \left( \frac{\kappa P_0 \nabla \delta P}{\rho} \right) - P_0 \nabla \cdot \mathbf{u}
\]

\[
\frac{m \partial \mathbf{v}}{\partial t} = g + a + \nabla P, \quad \text{Eq. (10)}
\]

where the substantial derivative has been replaced by the par-
tial derivative because \( u \nabla \delta P \ll P_0 \nabla \cdot \delta \mathbf{u} \). A rough estimate
using \( u \sim 10^{-2} \) m/s, \( \nabla \delta P \sim 10^{-3} \) Pa/m, \( P_0 = 10^5 \) Pa, and
\( \nabla \cdot \delta \mathbf{u} \sim 10^{-1} \) s\(^{-1} \) justifies this assumption. As in a hydro-
static system we assume that the pressure field of the magni-
ified system is \( \delta P^\prime(x^\prime, t) = \lambda \delta P(\lambda x, t) \), where \( x^\prime = \lambda x \). We
make the following scaling ansatz, expressing the solutions of
Eqs. (10) in terms of the scaled fields as

\[
\delta P(x, t) = \frac{1}{\lambda} \delta P^\prime(x^\prime, t)
\]

\[
\mathbf{u}_0 = \frac{1}{\lambda^2} \mathbf{u}_0^\prime
\]

\[
\delta \mathbf{u}(x, t) = \frac{1}{\lambda} \delta \mathbf{u}^\prime(x^\prime, t)
\]

\[
\rho(x, t) = \rho^\prime(x^\prime, t), \quad \text{Eq. (11)}
\]

where the sedimentation velocity \( \mathbf{u}_0^\prime \) scales as \( \lambda^2 \) because the
local density is \( \lambda \)-invariant and permeability goes as length
squared and scales as \( \lambda^3 \), i.e.

\[
\kappa^\prime = \lambda^2 \kappa. \quad \text{Eq. (12)}
\]

Note that Eqs. (10) are unaffected by the scaling of \( \mathbf{u}_0 \). Using
the new length scale in the derivative \( \nabla^\prime \) we get \( \nabla = \lambda \nabla^\prime \).
The pressure gradient \( \nabla \delta P, \rho, \phi \) and \( t \) are all invariant. By
substitution we obtain

\[
\frac{\phi}{\beta} \frac{\partial \delta P^\prime}{\partial t} = \nabla^\prime \cdot \left( \frac{\kappa^\prime \rho^\prime \nabla^\prime \delta P^\prime}{\rho^\prime} \right) - P_0^\prime \nabla^\prime \cdot \delta \mathbf{u}^\prime
\]

\[
\frac{1}{\lambda} \left( \frac{m \partial \mathbf{v}^\prime}{\partial t} - a^\prime \right) = g - \nabla^\prime \cdot \delta \mathbf{u}^\prime,
\]

as the equations satisfied by the scaled fields. Note that
by mass conservation of the granular phase, i.e. \( dp/dt = -\rho \nabla \cdot \delta \mathbf{u} \), the last term of the pressure equation may be writ-
ten \( -P_0 \nabla \cdot \delta \mathbf{u} = (P_0/\rho)dp/dt \), and this term thus gives the
effect of density changes in a frame of reference moving with
the grains. This implies that the term vanishes for a flow field
without internal compression or expansion. If such a flow field
is also steady, like that of a slab of connected particles mov-
ing at a constant velocity, the acceleration terms vanish along
with the \( -P_0 \nabla \cdot \delta \mathbf{u} \) term, i.e. all the \( \lambda \)-dependent terms van-
ish in Eq. (13). This invariance means that the scaled fields
are solutions of the same equations as the original fields.

The terms that are multiplied by a \( \lambda \)-factor break the invari-
ance, so what are their relative magnitude? To get an estimate
of this we introduce the dimensionless numbers. Taking the
characteristic length scale of the flow to be \( l \) and the time scale
relatively large. A likely that deviations from scaling behaviour arise due to the inertial term may be neglected since it is small compared to the scale invariance of low Reynolds number flows, where the invariant. However, since $F_r \propto D$. Gendron, H. Troadec, K. J. Måløy, and E. G. Flekkøy, Phys. Rev. E,

other hand, $A \propto 1/\lambda$ so decreasing the particle size will cause the compression term in the pressure equation to grow in relative magnitude. It is therefore only in a certain window of particle sizes around $d = 140 \mu m$ that we may expect the scale invariance. In particular, for the $d = 70 \mu m$ simulations it is likely that deviations from scaling behaviour arise due to the relatively large $A$-value.

The above scaling invariance is somewhat analogous to the scale invariance of low Reynolds number flows, where the inertial term may be neglected since it is small compared to the viscous term, and symmetries of the flow solutions emerge because the Navier Stokes equations are replaced by the Stokes equation.

VI. SUMMARY AND CONCLUSION

Experimental, numerical and theoretical investigations are performed in order to show the approximate size invariance of a granular flow instability when the spatial scale of the grains and the container are changed proportionally. The instability arises as a dense packing of grains, initially positioned above a gap of air, falls under gravity in a closed Hele-Shaw cell. Fine, equidistant fingers, defined by the grains as they fall, form along the grain-air interface. After an initial reduction in the number of fingers, or equivalently, an increase in the size of the structures, a steady state is reached where the structures maintain an approximately fixed size. The inverse characteristic size of the structures is measured by calculating the mean $\langle k \rangle$ of the solid fraction power spectrum. Seven simulations and four different experiments are performed where the spatial scale of the grains and the cells scale proportionally. Data-collapses of both the experimental and numerical data are obtained by plotting the rescaled mean wave number $d\langle k \rangle$, where $d$ is the grain diameter, as function of time. The conclusion is that the typical size of the structures that appear is invariant when measured in units of grain diameters. The numerical data are further analyzed and the wave number growth rates $\alpha_k$, valid for $t < 0.1$ seconds, are obtained from the $d = 140 \mu m$, 280 $\mu m$ and 490 $\mu m$ simulations. As the diameter increases the corresponding dispersion relations converge towards a functional form peaked around $dk = 0.02$.

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